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The viscous vortex induced by a sink on the axis of a circulating fluid in the presence of a plane free surface

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The flow under discussion represents an idealization of the bath-tub vortex; distortions of the free surface, finite sink size, and all rigid boundaries have been eliminated from the problem in order to isolate the effect of the non-uniform stretching of vortex lines produced by the sink flow. A boundary-layer type of approximation is made about the axis, which requires that the meridional Reynolds number (N) be large, and since the problem is still intractable, an expansion is made in powers of $K = R^2/N$ (where R is the swirl Reynolds number), which measures the strength of the interaction between the swirl and meridional velocity fields. In the limit of zero K the flow is a modified Burgers vortex whose radius decreases to zero at the sink. For non-zero K, the interaction is not restricted to the vortex core, because the presence of the vortex modifies the outer irrotational flow, inducing a radial mass flux into the core, whose dependence on the axial co-ordinate is calculated to the first order in K. The structure of the core is obtained, again to the first order in K, from two co-ordinate expansions, one near the stagnation point on the axis, and the other near the sink, although only the first few terms of the latter can be determined explicitly. It is shown how the methods can be extended not only to higher orders in K, but also to any other narrow viscous vortex in which the vortex lines are stretched non-uniformly away from an internal stagnation point.

1. Introduction

The problem to be discussed in this paper may be stated as follows. An infinite row of equally spaced point sinks, each of strength $4\pi Q$, is situated on the axis of an infinite body of viscous, incompressible fluid with uniform circulation $2\pi\Gamma_\infty$ at large radius (figure 1). The fluid withdrawn through the sinks is replaced irrotationally at infinity. The condition that the swirl velocity must be zero on the axis is violated by a potential vortex, so the flow must be rotational in some region surrounding the axis, and steady vortices will ultimately be set up in which the radial viscous diffusion of vorticity is balanced by the axial stretching of vortex lines produced by the flow into the sinks. We here examine both the internal structure of the vortices and their effect on the irrotational outer flow. By symmetry, we can restrict our attention to the single vortex in the region

$0 \leqslant z \leqslant h$, where $2h$ is the sink spacing and we use cylindrical polar co-ordinates (r, θ, z) with corresponding velocity components (u, v, w).

The problem is clearly allied to that of the familiar bath-tub vortex, but many complicating factors have been eliminated. These are: (*a*) distortion of the free surface resulting from reduced pressure near the axis: here the plane $z = 0$ is the equivalent of the free surface; (*b*) the finite size of any real sink; and (*c*) all rigid boundaries. Finite sink size will be unimportant far from the sink as long as the dimensions of the sink are small compared with a length-scale characteristic of the width of the vortex near the stagnation point O (figure 1). Rigid boundaries,

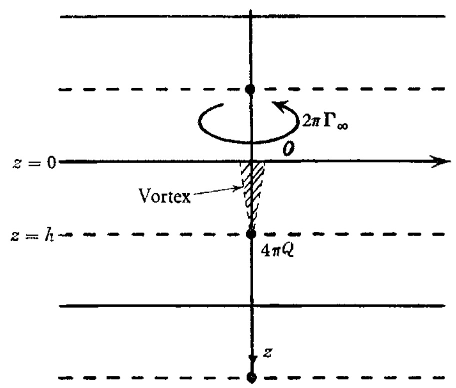

FIGURE 1. Pictorial statement of the problem.

however, always have a significant effect on vortex motions (Rott & Lewellen 1966, chapter v), and their neglect is an idealization which cannot be achieved in practice, but which is justifiable here because it isolates a hitherto ignored aspect of real vortex flows, the non-uniform stretching of vortex lines. The equation governing the axial component of vorticity, ω, in a steady, axisymmetric situation, is

$$u\frac{\partial \omega}{\partial r} + w\frac{\partial \omega}{\partial z} = \omega\frac{\partial w}{\partial z} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial r} + \nu\nabla^2\omega, \tag{1.1}$$

where ν is the kinematic viscosity of the fluid. If u is a function of r alone, and hence (from the continuity equation) $\partial w/\partial z$ is independent of z, then it is possible for ω, and therefore v, also to be independent of z. In that case the only effect of the swirl on the meridional flow field (through the other equations of motion) is to alter the pressure distribution. Thus only if $\partial w/\partial z$, representing the rate of stretching of vortex lines, is non-uniform, i.e. if it depends on z, can there be any interaction between the swirl velocity field (determined by (1.1)) and the meridional flow field.

In many previous models of viscous vortices the rate of stretching of vortex lines is uniform, v depends only on r, and there is no interaction (see, for example, Donaldson & Sullivan 1960). This is typified by the well-known stagnation-point vortex first described by Burgers (1940) in which

$$\left.\begin{aligned} u &= -kr, \quad w = 2kz, \\ v &= \frac{\Gamma_\infty}{r}\left\{1 - \exp\left(\frac{-kr^2}{2\nu}\right)\right\}, \end{aligned}\right\} \tag{1.2}$$

where k is a constant and $2\pi\Gamma_\infty$ is a uniform circulation at large radius. The meridional velocity field (u, w) is that of irrotational, axisymmetric stagnation point flow away from the plane surface $z = 0$, and (1.2) satisfies both the full Navier–Stokes equations and all boundary conditions on the axis. In the sink flow considered here, however, $\partial w/\partial z$ on the axis (say) is far from uniform (w is zero at $z = 0$ and infinite at $z = h$), and so there must be a significant interaction between the sink flow and the swirl.

There are other studies of viscous vortices in which the stretching of vortex lines is non-uniform, but these often take the form of similarity solutions in which $\partial w/\partial z$ varies as a given power of z, see, for example, Long (1958, 1961), or Lewellen (1964). The only work known to the author where a general dependence on z is permitted is by Lewellen (1962), who makes an expansion for vortices with strong swirl (v is an order of magnitude greater than w), and can obtain determinate solutions only if the stream function is completely known as a function of radius at two axial stations. Direct application of this method in the present instance leads to a purely irrotational flow which is singular at all points on the axis, and tells us nothing of the viscous vortex structure, even in the case of strong swirl.

Idealized as it is, the problem under discussion is still intractable, and some further simplifications must be made. The most obvious is to assume that the width of the rotational vortex core (characterized by a length δ, say) is small compared with the axial length scale h. Thus we make a boundary-layer type of approximation about the axis, and this turns out to require that the Reynolds number of the basic sink flow ($N = Q/\nu h$) be large. One might then suppose, by analogy with first-order boundary-layer theory, that the flow outside the core will be the irrotational flow with uniform circulation which would exist in an inviscid fluid, and that the limit of this flow, as the outer variable r/h tends to zero, must be imposed as the limit of the core flow as the inner variable r/δ tends to infinity (the matching condition). However, that supposition would conflict with known experimental results for vortices with strong swirl, which indicate that the irrotational outer flow consists of a uniform radial inflow ($ru =$ constant, $w = 0$) superimposed on the uniform circulation (Long 1958; Turner 1966; such an outer flow is often assumed in theoretical models, e.g. by Lewellen 1962, 1964). The irrotational flow due to a row of point sinks is quite different, since u tends to zero at the axis and w tends to a function of z. We are thus forced to conclude that the boundary-layer analogy breaks down and that the presence of the vortex has a modifying effect on the outer flow, for which we cannot assume a given form, but can only hope that the matching condition will serve to determine it.

Clearly, the outer flow will vary, from the strong swirl limit of uniform radial inflow on the one hand, to the undisturbed irrotational sink flow (which would exist in the absence of swirl) on the other. It has already been mentioned that known methods for the strong swirl limit do not yield a realistic solution, and the general problem is still too difficult, so in this paper we consider the limit of weak swirl. The method is to assume small perturbations from the irrotational no-swirl solution and to expand in powers of a small parameter K ($K = \Gamma_\infty^2 h/Q\nu$

and is here referred to as the interaction parameter). Lewellen (1965) also used the idea of linearizing about flows with no swirl, but the example he chose was pure stagnation point flow, in which the rate of stretching of vortex lines is uniform, and which is thus of little interest. The basic irrotational flow in this problem is the sink flow given by the Stokes stream function

$$\psi_0 = Q \sum_{n=0}^{\infty} \left(\frac{(2n+1)h-z}{\{r^2+[(2n+1)h-z]^2\}^{\frac{1}{2}}} - \frac{(2n+1)h+z}{\{r^2+[(2n+1)h+z]^2\}^{\frac{1}{2}}} \right) \tag{1.3}$$

and the limit of this as r/h tends to zero (the inner limit of the zero-order outer solution) is

$$\psi_0 \bigg|_{r/h \to 0} = -2Qr^2zh \sum_{n=0}^{\infty} \frac{2n+1}{[(2n+1)^2h^2-z^2]^2}. \tag{1.4}$$

Note that as we approach the stagnation point ($z/h \to 0$), (1.4) becomes the stream function for pure stagnation point flow, with k (in 1.2) equal to

$$\frac{2Q}{h^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \approx (1{\cdot}05)\frac{2Q}{h^3}.$$

Thus if the vortex in any way resembles a Burgers vortex at the stagnation point, which seems likely, the radial length scale is given by (1.2) as

$$\delta^2 = 2\nu/k \approx \nu h^3/Q. \tag{1.5}$$

This radial length scale will form the basis for the non-dimensionalization in the next section.

In § 3, the solution for the circulation field is examined to the zeroth-order in K, and, to this order, the flow *is* seen to have the form of a Burgers vortex whose radius varies from δ at the stagnation point to zero at the sink.

The first-order solution is investigated in § 4, and it is shown that the outer flow is modified by the presence of the vortex, which induces a radial inflow to the core near the stagnation point, and a corresponding outflow near the sink. The magnitude of this inflow as a function of z is derived explicitly.

In §§ 5 and 6, the detailed structure of the vortex core is examined, to the first order in K, by means of two co-ordinate expansions, in powers of z/h and $(1-z/h)$ respectively. The former is completely determinate, and the first few terms, giving the radial dependence of the flow variables near the stagnation point, are computed. Despite the lack of a mathematical proof of convergence, it is argued that these few terms give a reasonable picture of the flow for values of z/h less than about 0·4. The second expansion, near the sink, is not completely determinate, because there are no means by which the boundary conditions on $z = 0$ can be incorporated into it. However, the leading terms can be calculated explicitly and reveal something of the flow structure near the sink.

The second-order (K^2) solution is briefly discussed in § 7, and the first terms in the z/h expansion are calculated. The smallness of these terms compared with the corresponding terms of the first-order solution is adduced as evidence that the small K expansion is a good asymptotic representation of the complete solution. Since, in addition, the present methods are suitable for a whole class of vortices, of which ours is but one example, it seems clear that they lead to results which are valid in a wide range of circumstances.

2. Non-dimensional equations and boundary conditions

The equations of motion for steady axisymmetric flow with swirl, in the chosen co-ordinate system (r, θ, z), can be reduced to two in number by eliminating the pressure and by expressing the radial and axial velocity components in terms of a Stokes stream function $\hat{\psi}$ as follows:

$$u = (1/r)\,\hat{\psi}_z, \quad w = -(1/r)\,\hat{\psi}_r, \tag{2.1}$$

where the subscript denotes differentiation with respect to the relevant variable. The dimensional parameters which govern the problem are the sink strength $4\pi Q$, the circulation at infinity $2\pi\Gamma_\infty$, the sink spacing $2h$, and the kinematic viscosity ν. The obvious axial length scale is h, and the obvious scale for the circulation rv is Γ_∞, but the radial length scale and the scale for the stream function will differ according to which of the two flow regions, the inner viscous core or the outer region of presumably irrotational flow, is being considered.

(i) *The inner region*

Here we non-dimensionalize the equations by means of the following transformations

$$\xi = z/h, \quad \eta = r^2/\delta^2, \quad \Gamma = rv/\Gamma_\infty, \quad \psi = \hat{\psi}/\nu h, \tag{2.2}$$

where δ is defined by (1.5), and $\hat{\psi}$ is non-dimensionalized so that the inner solution without swirl, (1.4), has dimensionless form. The velocity components are now given in terms of the non-dimensional variables by

$$ru = \nu\psi_\xi, \quad w = -\frac{2\nu h}{\delta^2}\psi_\eta, \quad rv = \Gamma_\infty\Gamma, \tag{2.3}$$

and the two equations of motion become (cf. Lewellen 1962)

$$\psi_\xi\Gamma_\eta - \psi_\eta\Gamma_\xi = 2\eta\Gamma_{\eta\eta} + \tfrac{1}{2}\alpha^2\Gamma_{\xi\xi} \tag{2.4}$$

and
$$\begin{aligned} K\Gamma\Gamma_\xi = {} & 4\eta^2(\psi_\xi\psi_{\eta\eta\eta} - \psi_\eta\psi_{\xi\eta\eta} - 4\psi_{\eta\eta\eta} - 2\eta\psi_{\eta\eta\eta\eta}) \\ & + \alpha^2(-\psi_\xi\psi_{\xi\xi} + \eta\psi_\xi\psi_{\xi\xi\eta} - \eta\psi_\eta\psi_{\xi\xi\xi} - 4\eta^2\psi_{\xi\xi\eta\eta} - \tfrac{1}{2}\alpha^2\eta\psi_{\xi\xi\xi\xi}), \end{aligned} \tag{2.5}$$

where
$$\alpha^2 = \delta^2/h^2 = \nu h/Q = 1/N,$$
$$K = \Gamma_\infty^2 h/Q\nu = R^2/N,$$

and N, R are the radial and tangential Reynolds numbers respectively. (The reason for calling K the interaction parameter is clear from (2.5): if K is zero, ψ is determined by the same equation as in the absence of swirl, but as K increases the coupling between the two equations, and hence the interaction between the swirl and meridional flow fields, becomes more pronounced.) If we now make the boundary-layer approximation ($\alpha^2 \ll 1$ or $N \gg 1$), the equations are greatly simplified, and become

$$\psi_\xi\Gamma_\eta - \psi_\eta\Gamma_\xi = 2\eta\Gamma_{\eta\eta} \tag{2.6}$$

and
$$K\Gamma\Gamma_\xi = 4\eta^2(\psi_\xi\psi_{\eta\eta\eta} - \psi_\eta\psi_{\xi\eta\eta} - 4\psi_{\eta\eta\eta} - 2\eta\psi_{\eta\eta\eta\eta}). \tag{2.7}$$

The boundary conditions to be imposed on the inner variables are:

(a) on $z = 0$, and on $z = h$ for $r > 0$, the axial velocity and the tangential stress (by symmetry) are zero, i.e.

$$\left.\begin{aligned} \text{on}\quad \xi = 0 \quad (\eta \geqslant 0) \\ \xi = 1 \quad (\eta > 0) \end{aligned}\right\} \psi_\eta = \psi_{\xi\xi} = \Gamma_\xi = 0; \tag{2.8}$$

(b) on $r = 0$, the radial and swirl velocities are zero, and all components of shear stress are zero, i.e. as

$$\eta \to 0 \quad (0 \leqslant \xi < 1), \quad \psi \sim (\text{function of } \xi)\,\eta, \quad \Gamma \sim (\text{function of } \xi)\,\eta; \tag{2.9}$$

(c) as r/δ tends to infinity, the solution must match the outer flow, i.e. as

$$\eta \to \infty \quad (0 \leqslant \xi \leqslant 1), \quad \psi, \Gamma \sim \lim_{r/h \to 0} \ (\text{outer solution}). \tag{2.10}$$

Note that when K is zero (no swirl), the inner solution for $\hat{\psi}$ is the unmodified inner limit of the irrotational solution, and is given by (1.4). In other words as $K \to 0$,

$$\psi \sim \psi_0 = -2\xi\eta \sum_{n=0}^{\infty} \frac{2n+1}{[(2n+1)^2 - \xi^2]^2} = -2\xi\eta F(\xi) \quad (\text{say}). \tag{2.11}$$

The function $F(\xi)$ is here introduced purely for convenience, but most of what follows is perfectly valid for any even function $F(\xi)$ which is bounded at $\xi = 0$. Thus the methods of this paper may be used for many possible vortices, not only the particular sink vortex with which we are at present concerned.

(ii) *The outer region*

In this region, where we expect viscous forces to be unimportant, there is a single length scale h, and the stream function should be non-dimensionalized so that the condition at large radius (uniform inflow) is non-dimensional. Here, therefore, we use the transformations

$$\xi = z/h, \quad \rho = r/h, \quad \tilde{\Gamma} = ru/\Gamma_\infty, \quad \tilde{\psi} = \hat{\psi}/Q, \tag{2.12}$$

and in the resulting equations we neglect the viscous terms, which are all multiplied by a factor $1/N$. The equations are satisfied if the flow is irrotational, that is, if the circulation is uniform and the azimuthal vorticity component is zero. Thus

$$\tilde{\Gamma} \equiv 1$$

and

$$\tilde{\psi}_{\rho\rho} - (1/\rho)\,\tilde{\psi}_\rho + \tilde{\psi}_{\xi\xi} = 0. \tag{2.13}$$

The boundary conditions on $\tilde{\psi}$ are:

(a)
$$\left.\begin{aligned} \text{on}\quad \xi = 0 \quad (\rho \geqslant 0) \\ \xi = 1 \quad (\rho > 0) \end{aligned}\right\} \tilde{\psi}_\rho = \tilde{\psi}_{\xi\xi} = 0; \tag{2.14}$$

(b)
$$\text{as}\quad \rho \to \infty \quad (0 \leqslant \xi \leqslant 1) \quad \tilde{\psi} \sim -\xi. \tag{2.15}$$

The inner boundary condition on $\tilde{\psi}$ is given by the matching condition (2.10)

$$\psi(\xi, \eta \to \infty) = N\tilde{\psi}(\xi, \rho \to 0). \tag{2.16}$$

Also from (2.10), the matching condition on Γ is seen to be

$$\Gamma \to 1 \quad \text{as} \quad \eta \to \infty. \tag{2.17}$$

Finally, from (1.3), the limit of $\tilde{\psi}$ as K tends to zero is given by

$$\tilde{\psi}_0 = \sum_{n=0}^{\infty} \left\{ \frac{2n+1-\xi}{[\rho^2+(2n+1-\xi)^2]^{\frac{1}{2}}} - \frac{2n+1+\xi}{[\rho^2+(2n+1+\xi)^2]^{\frac{1}{2}}} \right\}. \tag{2.18}$$

3. The small K expansion. Zero-order inner solution

As indicated in the introduction, a solution to the problem for general values of K has not proved feasible, and in order to proceed further we must restrict ourselves to conditions of weak swirl and expand in powers of K. The expansions for the inner variables are written

$$\left.\begin{aligned} \psi(K,\xi,\eta) &= \psi_0(\xi,\eta)+K\psi_1(\xi,\eta)+K^2\psi_2(\xi,\eta)+\ldots, \\ \Gamma(K,\xi,\eta) &= \Gamma_0(\xi,\eta)+K\Gamma_1(\xi,\eta)+K^2\Gamma_2(\xi,\eta)+\ldots, \end{aligned}\right\} \tag{3.1}$$

(the expansion for the outer variable $\tilde{\psi}(K,\xi,\rho)$ is exactly similar), where the leading term in the expansion for ψ is the inner irrotational solution (2.11)

$$\psi_0 = -2\xi\eta F(\xi).$$

This leading term suggests the following change of independent inner variables

$$x = \xi, \quad y = \eta F(\xi), \tag{3.2}$$

in terms of which the equations (2.6) and (2.7) become

$$\psi_x \Gamma_y - \psi_y \Gamma_x = 2y\Gamma_{yy} \tag{3.3}$$

and

$$\frac{K}{4y^2F(x)}\Gamma\left[\Gamma_x+\frac{yF'(x)}{F(x)}\Gamma_y\right] = \psi_x\psi_{yyy}-\psi_y\psi_{xyy}-\frac{2F'(x)}{F(x)}\psi_y\psi_{yy}-4\psi_{yyy}-2y\psi_{yyyy}. \tag{3.4}$$

Note that the transformation (3.2) is singular at $\xi = 1$, because $F(\xi)$ is unbounded there, so that the limit $y \to \infty$, corresponding to the limit $\eta \to \infty$ for $0 \leqslant \xi < 1$, corresponds to any positive η for $\xi = 1$.

We may now substitute the expansion (3.1) with (x, y) replacing (ξ, η) and with

$$\psi_0 \equiv -2xy, \tag{3.5}$$

into equations (3.3) and (3.4), and equate like powers of K. The leading terms (K^0) in (3.3), with the use of (3.5), yield the following equation for $\Gamma_0(x,y)$

$$x\Gamma_{0x}-y\Gamma_{0y} = y\Gamma_{0yy}. \tag{3.6}$$

The boundary conditions on $\Gamma_0(x,y)$ are

$$\left.\begin{aligned} \Gamma_{0x}(0,y) = 0, \quad &\text{from (2.8)}; \\ \Gamma_0(x,y\to 0) \sim y(\text{function of } x), \quad &\text{from (2.9)}; \\ \Gamma_0(x,\infty) = 1, \quad &\text{from (2.17)}. \end{aligned}\right\} \tag{3.7}$$

If we seek a separable solution to (3.6), we find that it must be of the form $x^\beta g(y)$ where β is a constant which must be either zero or greater than one for the first of conditions (3.7) to be satisfied. But if β is greater than one, no solution of the resulting equation for $g(y)$ is both regular at $y = 0$ and bounded as $y \to \infty$. Hence the only separable solution to (3.6) which can satisfy all the boundary conditions is independent of x, and can easily be seen to be

$$\Gamma_0(x, y) \equiv 1 - e^{-y}. \tag{3.8}$$

Now the dimensionless form of the Burgers vortex (1.2) is

$$\psi = -2\xi\eta, \quad \Gamma = 1 - e^{-\eta}$$

(when $k = 2Q/h^3$). Hence (3.5) and (3.8) demonstrate that for asymptotically small values of K the flow in our sink vortex is essentially a Burgers vortex with a variable radial length scale, say $\Delta(z)$, which is given in terms of δ by

$$[\Delta(z)/\delta]^2 = 1/F(z/h). \tag{3.9}$$

The function $F(\xi)$, given by (2.11), increases monotonically from approximately unity at $\xi = 0$ to infinity at $\xi = 1$, so the radius of our modified Burgers vortex shrinks monotonically from approximately δ at the stagnation point to zero at the sink. This indicates both that the inner non-dimensionalization of § 2 (i) was correct, and that the boundary-layer approximation is valid even in the neighbourhood of the sink, where it might have been expected to break down. A final point to notice is that for non-zero values of K, the circulation Γ will tend to the form of a Burgers vortex as the stagnation point is approached *only* if all the functions $\Gamma_n(x, y)$, $n > 0$, tend to zero as x tends to zero. Otherwise the flow differs from a Burgers vortex, even near the stagnation point.

4. First-order solution. Radial inflow to the core

We now consider the leading term (K^1) in (3.4), which leads to an equation for ψ_1. Using (3.5) and (3.8), we obtain

$$y\psi_{1yyyy} + (y+2)\psi_{1yyy} - x\psi_{1xyy} - \frac{2xF'(x)}{F(x)}\psi_{1yy} = -\frac{F'(x)}{8F^2(x)}\frac{e^{-y} - e^{-2y}}{y}. \tag{4.1}$$

This equation can be integrated twice with respect to y, giving

$$\begin{aligned} &y\psi_{1yy} + y\psi_{1y} - 2\psi_1\left[1 + \frac{xF'(x)}{F(x)}\right] - x\psi_{1x} \\ &\quad = \frac{F'(x)}{8F^2(x)}\{y[E_1(y) - E_1(2y)] - e^{-y} + \tfrac{1}{2}e^{-2y}\} + G(x) + yH(x), \end{aligned} \tag{4.2}$$

where $E_1(y)$ is the exponential integral

$$E_1(y) = \int_y^\infty \frac{e^{-t}}{t}\,dt$$

and $G(x)$, $H(x)$ are functions of integration. In order for ψ_1 to satisfy the boundary condition $\psi_1 \sim y$(function of x) as $y \to 0$, obtained from (2.9), the function $G(x)$ cannot be identically zero, and must be given by

$$G(x) \equiv \frac{F'(x)}{16F^2(x)}. \tag{4.3}$$

Now, if the outer flow were described by the undisturbed irrotational stream function $\tilde{\psi}_0$ (given in (2.18)), then the functions ψ_1, ψ_2, etc. would all tend to zero as y tends to infinity, from the matching condition (2.16). But (4.2) shows that ψ_1 cannot tend to zero as y tends to infinity, even if $H(x)$ is identically zero, because $G(x)$ is non-zero. Hence the outer flow is modified. Assuming that $H(x)$ *is* identically zero, by the 'principle of minimum singularity' (Van Dyke 1964, p. 53; this principle can be justified by a consideration of the next term, of order $1/N$, in the boundary-layer expansion), we see from (4.2) that as y tends to infinity, $\psi_1(x,y)$ asymptotically becomes a function of x alone, say

$$\psi_1(x,y) \sim \phi_1(x) \quad \text{as} \quad y \to \infty.$$

The asymptotic form of (4.2) is then

$$x\phi_1' + 2\left(1+\frac{xF'}{F}\right)\phi_1 = -G = -\frac{F'}{16F^2}, \tag{4.4}$$

where a prime denotes differentiation with respect to x. Equation (4.4) must be solved subject to the condition that $\phi_1(0) = 0$, and the solution is

$$\phi_1(x) = \frac{-\int_0^x F'(t)\,t\,dt}{16x^2F^2(x)}, \tag{4.5}$$

where $F(x)$, we recall, is given by (2.11). $\phi_1(x)$ is the value of the stream function ψ at the edge of the core, to the first order in K, and demonstrates that there is a non-zero radial mass flow (ru) at the edge of the core, given by

$$ru = \nu\psi_\xi = \nu\phi_1'(x).$$

The function $\phi_1'(x)$ is plotted in figure 2. It can be seen that the vortex induces a radial inflow to the core near the stagnation point, and (by conservation of mass) a corresponding outflow nearer the sink.

From our knowledge of $\phi_1(x)$ we can calculate the first-order modification to the outer flow, represented by the stream function $\tilde{\psi}_1(\xi,\rho)$. This function must satisfy the equation of irrotationality (2.13), and the boundary conditions

$$\left.\begin{aligned} \tilde{\psi}_{1\rho} = \tilde{\psi}_{1\xi\xi} = 0 \quad &\text{on} \quad \xi = 0 \quad \text{and} \quad \xi = 1 \quad \text{from (2.14);} \\ \tilde{\psi}_1 \to 0 \quad &\text{as} \quad \rho \to \infty \quad \text{from (2.15);} \\ \tilde{\psi}_1 \sim (1/N)\,\phi_1(\xi) \quad &\text{as} \quad \rho \to 0 \quad \text{from (2.16).} \end{aligned}\right\} \tag{4.6}$$

The general solution of (2.13) satisfying the first two of the conditions (4.6) is

$$\tilde{\psi}_1 = \frac{1}{N}\sum_{m=1}^{\infty} A_m m\pi\rho K_1(m\pi\rho)\sin(m\pi\xi), \tag{4.7}$$

where K_1 is the first-order modified Bessel function of the second kind, and where the constants A_m are determined from the third of the conditions (4.6) to be

$$A_m = 2\int_0^1 \phi_1(\xi)\sin(m\pi\xi)\,d\xi. \tag{4.8}$$

These constants have been evaluated on a computer for values of m up to 55; the magnitude of A_m decreases as m increases, and A_{m+1} has the opposite sign to A_m for all $m \geqslant 2$ (some of the terms are given in table 1). It seems, therefore, that the series (4.7) converges rapidly. The streamlines of this perturbation to the outer flow have been calculated from (4.7), and are plotted in figure 3(*b*). They must be regarded as superimposed onto the undisturbed streamline pattern of figure 3(*a*). The perturbation is very small for small K and large N.

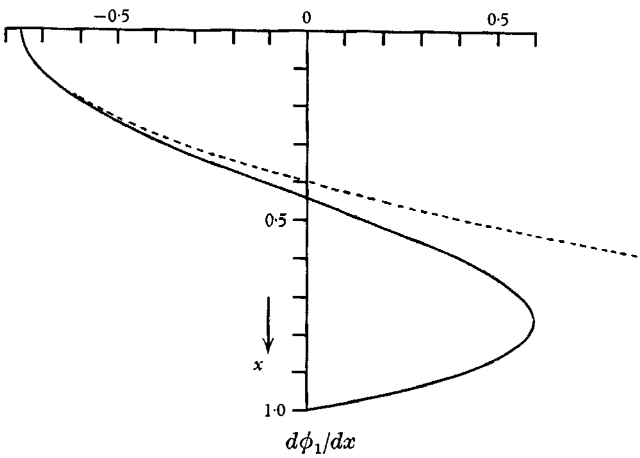

FIGURE 2. The radial mass flow at the edge of the vortex core, as a function of axial distance: $\phi_1'(x)$. The broken line is the approximation to $\phi_1'(x)$ obtained from the first two terms of an expansion in powers of x.

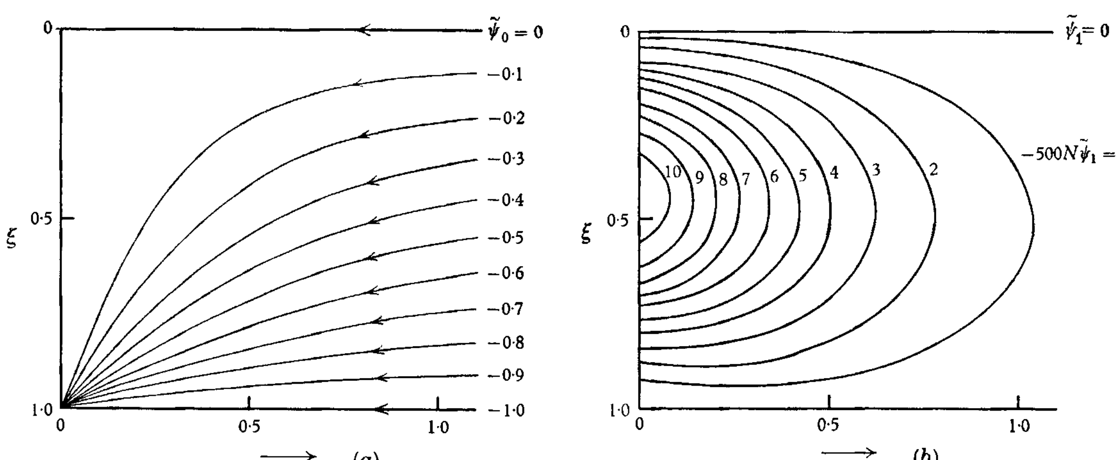

FIGURE 3. Streamlines (*a*) of the undisturbed outer flow, and (*b*) of the perturbation to it induced by the radial mass flow of figure 2.

m	A_m
1	$-1{\cdot}99\times10^{-2}$
2	$-3{\cdot}29\times10^{-3}$
3	$+1{\cdot}26\times10^{-3}$
4	$-5{\cdot}44\times10^{-4}$
5	$+2{\cdot}76\times10^{-4}$
6	$-1{\cdot}58\times10^{-4}$
...	....
50	$-1{\cdot}88\times10^{-7}$

TABLE 1. Some coefficients of the Fourier series (4.7)

5. First-order inner solution near the stagnation point

In order to examine the actual structure of the flow in the core, to the first order in K, we must first solve equation (4.2) for ψ_1. Then, knowing ψ_1, we can in principle also find Γ_1, for the equation for Γ_1 comes from the first-order (K^1) terms in (3.3), and, using (3.5) and (3.8), it is

$$y\Gamma_{1yy}+y\Gamma_{1y}-x\Gamma_{1x} = \tfrac{1}{2}e^{-y}\psi_{1x}. \tag{5.1}$$

There is no closed form solution for ψ_1 or Γ_1, in general (see the end of this section for an exception), and the only way to reduce equations (4.2) or (5.1) to a set of ordinary differential equations seems to be to make a formal expansion in powers of the axial co-ordinate. We expand in this section about the stagnation point ($x = 0$), and in the next section about the sink ($x = 1$).

Formally, therefore, let us expand ψ_1 and Γ_1 in powers of x. Making use of the conditions $\psi_{1y} = \psi_{1xx} = \Gamma_{1x} = 0$ on $x = 0$ (from (2.8), since $F'(0) = 0$), we may write

$$\left.\begin{aligned}\psi_1 &= x\psi_{11}(y)+x^3\psi_{13}(y)+\ldots,\\ \Gamma_1 &= \Gamma_{10}(y)+x^2\Gamma_{12}(y)+\ldots.\end{aligned}\right\} \tag{5.2}$$

In fact, by symmetry, Γ_1 and ψ_{1x} must both be even functions of x (as $F(x)$ is an even function), so that *all* the odd powers disappear from Γ_1 and *all* the even powers disappear from ψ_1; this would emerge from the equations if it were not assumed. We shall also need the expansion of $F(x)$, which may be written

$$F(x) = a_0(1+a_1x^2+a_2x^4+\ldots) \tag{5.3}$$

so that the method may be generalized beyond the present case, in which, from (2.11),

$$a_0 = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} \approx 1{\cdot}052, \quad a_1 = \frac{2}{a_0}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5} \approx 1{\cdot}905, \quad \text{etc.} \tag{5.4}$$

The expansion for $G(x)$, from (4.3), is

$$G(x) = (x/8a_0)\,[a_1+2x^2(a_2-a_1^2)+\ldots]. \tag{5.5}$$

The procedure now is to substitute (5.2) and (5.5) into (4.2) and (5.1), equate like powers of x, and solve for each function in turn. The first power of x in (4.2) gives the equation for $\psi_{11}(y)$, as follows (here a prime denotes differentiation with respect to y):

$$y\psi_{11}''+y\psi_{11}'-3\psi_{11} = (a_1/4a_0)\,D_1(y), \tag{5.6}$$

where $$D_1(y) \equiv y[E_1(y) - E_1(2y)] - e^{-y} + \tfrac{1}{2}e^{-2y} + \tfrac{1}{2}.$$

The next power of x (x^3) in (4.2) gives the equation for $\psi_{13}(y)$, and so on. Similarly the leading term in (5.1) gives, for $\Gamma_{10}(y)$:

$$y\Gamma''_{10} + y\Gamma'_{10} = \tfrac{1}{2}e^{-y}\psi_{11}(y), \tag{5.7}$$

and the term in x^2 gives the equation for $\Gamma_{12}(y)$, and so on. The boundary conditions on the ψ_{1n} and Γ_{1n} are

$$\psi_{1n}, \Gamma_{1n} \propto y \quad \text{as} \quad y \to 0, \quad \text{and} \quad \psi'_{1n}, \quad \Gamma_{1n} \to 0 \quad \text{as} \quad y \to \infty.$$

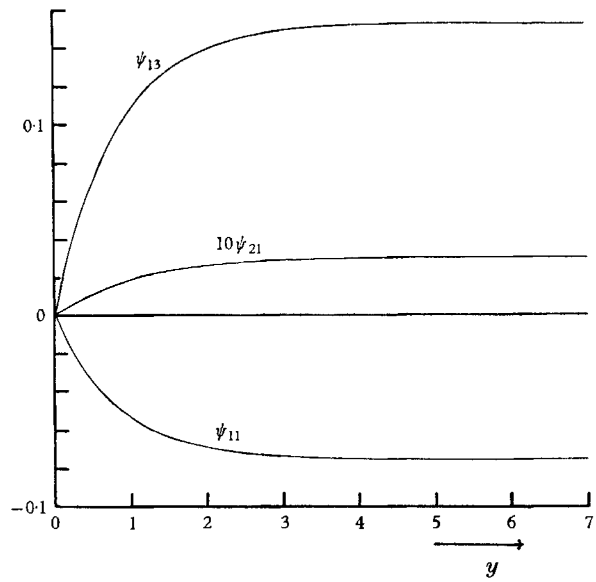

FIGURE 4. The leading terms of the stream function expansions near $x = 0$: $\psi_{11}(y)$, $\psi_{13}(y)$ and $\psi_{21}(y)$.

The solution of (5.6) satisfying these conditions is

$$\begin{aligned}\psi_{11}(y) = (a_1/8a_0)\{&[E_1(y) - E_1(2y)]\left(\tfrac{2}{3}y^3 + 4y^2 + 3y\right) \\ &- \tfrac{1}{3}e^{-y}(2y^2 + 10y + 1) + \tfrac{1}{6}e^{-2y}(2y^2 + 11y + 4) - \tfrac{1}{3}\},\end{aligned} \tag{5.8}$$

and the solution of (5.7) could doubtless also be expressed in closed form, but it is more convenient to solve this and subsequent equations numerically (using a standard Runge–Kutta technique). The graphs of $\psi_{11}(y)$ and $\psi_{13}(y)$ are given in figure 4 (with a_0, a_1 given by (5.4)); those of $\psi'_{11}(y)$ and $\psi'_{13}(y)$, which are of interest because the axial velocity w is proportional to $-F(x)\psi_y$ from (2.3) and (3.2), are given in figure 5, and those of $\Gamma_{10}(y)$ and $\Gamma_{12}(y)$ are given in figure 6. Discussion of these and similar results will be postponed to § 8, but we may notice here that $\Gamma_{10}(y)$ is not identically zero, and so the flow does not tend to that of a Burgers vortex as $x \to 0$, for non-zero K.

Finally, it should be noticed that (4.2) does have a self-similar solution for the particular case where $$F(x) \equiv c_0(1 + c_1x^q).$$

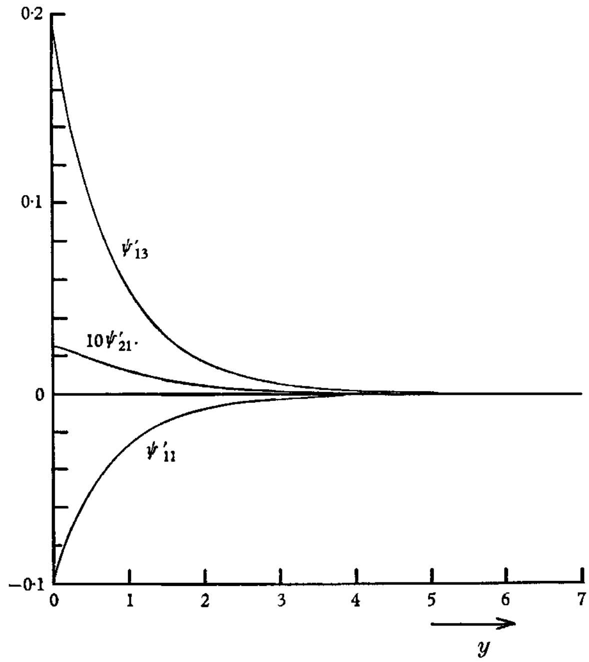

FIGURE 5. The functions $\psi'_{11}(y)$, $\psi'_{13}(y)$ and $\psi'_{21}(y)$.

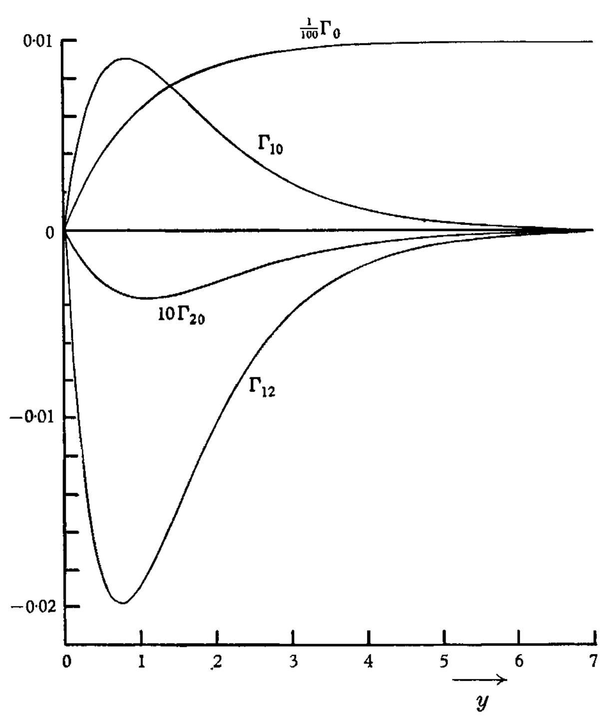

FIGURE 6. The leading terms of the circulation expansions near $x = 0$: $\Gamma_0(y)$, $\Gamma_{10}(y)$, $\Gamma_{12}(y)$ and $\Gamma_{20}(y)$.

In that case we can write
$$\psi_1 = f(y)\,g(x),$$
where
$$g(x) \equiv x^{q-1}/F^2(x)$$
and $f(y)$ satisfies
$$yf'' + yf' - (q+1)f = \tfrac{1}{8}qc_0c_1D_1(y), \tag{5.9}$$
which has the same form as (5.6), and may be solved similarly. If $q = 2$, then $f(y)$ is the same as $\psi_{11}(y)$ in (5.8) with a_1/a_0 replaced by qc_0c_1. Even in this case, however, there is no similarity solution of (5.1) for Γ_1 unless $q = 0$, which reduces the problem to the uninteresting one of a cylindrical vortex.

6. First-order inner solution near the sink

The sink ($\xi = 1, \eta = 0$) is a singular point of the flow field, and in particular of the function $F(\xi)$. Hence it is improbable that a regular Taylor expansion like (5.2) can represent the solution there. Other possibilities must be considered, in particular, from previous experience with singular expansions (Van Dyke 1964, p. 200), logarithmic terms might be expected to appear. In terms of $\bar{x} = 1-x$ (4.2) for ψ_1 is
$$y\psi_{1yy} + y\psi_{1y} - 2\psi_1 + 2(1-\bar{x})\frac{\bar{F}'(\bar{x})}{\bar{F}(\bar{x})}\psi_1 + (1-\bar{x})\,\psi_{1\bar{x}} = -\frac{\bar{F}'(\bar{x})}{8\bar{F}^2(\bar{x})}D_1(y), \tag{6.1}$$
where $D_1(y)$ is given by (5.6), and from (2.11)
$$\bar{F}(\bar{x}) \equiv F(x) = (1/4\bar{x}^2)\,[1+\bar{x}+4b_0\bar{x}^2+O(\bar{x}^3)] \tag{6.2}$$
with
$$b_0 = \frac{3}{16} + \sum_{n=1}^{\infty}\frac{2n+1}{16n^2(n+1)^2} = \frac{1}{4}. \tag{6.3}$$

The leading term on the right-hand side of (6.1) is of order $\bar{x}$, so (since $\bar{F}'/\bar{F}$ is of order $1/\bar{x}$) the leading term in the expansion for ψ_1 must be of order $\bar{x}^2$. If we assumed a Taylor expansion, starting with the $\bar{x}^2$ term, and proceed as in § 5, we discover an inconsistency at order $\bar{x}^3$ in (6.1), which can be eliminated only by the introduction of logarithms. Let us, therefore, expand ψ_1 as follows:
$$\psi_1 = \bar{x}^2f_{12}(y) + \bar{x}^3f_{13}(y) + \bar{x}^4f_{14}(y) + \ldots + \log\bar{x}\,[\bar{x}^4f_{L4}(y) + \bar{x}^5f_{L5}(y) + \ldots]. \tag{6.4}$$
Substituting (6.2) and (6.4) into (6.1), and equating like powers of $\bar{x}$, we obtain a series of equations which yield explicit expressions for the functions f_{12}, f_{13}, f_{14}, f_{15} [for example, the term in $\bar{x}$ gives $f_{12} = -\frac{1}{2}D_1(y)$], which, with their first derivatives, are plotted in figures 7 and 8.

The difficulty with extending these solutions to higher powers of $\bar{x}$ lies not in the complexity of the functions, but in the fact that the function f_{14} cannot be computed, since the equation which should determine it (the $\bar{x}^3\log\bar{x}$ term in (6.1)) reduces to the trivial $0 = 0$. The subsequent functions f_{1n} themselves depend on f_{14}. It is not surprising that the expansion near the sink cannot generate the complete solution for ψ_1, because it takes no account of the boundary conditions on $x = 0$ (the reverse process *can* work (§ 5), because the boundary conditions at the sink itself ($x = 1$, $y < \infty$) are arbitrary). A determination of $f_{14}(y)$, and hence of the complete small $\bar{x}$ expansion, depends on a knowledge of the

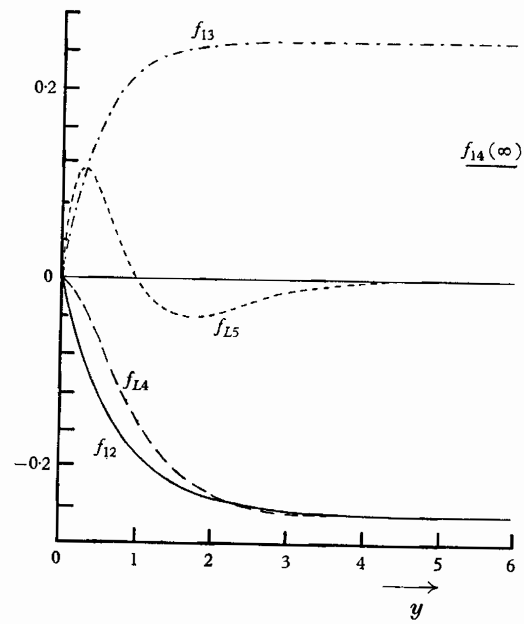

FIGURE 7. The leading terms of the first-order stream function expansion near $x = 1$: $f_{12}(y)$, $f_{13}(y)$, $f_{L4}(y)$, $f_{L5}(y)$. Note also the value of $f_{14}(\infty)$.

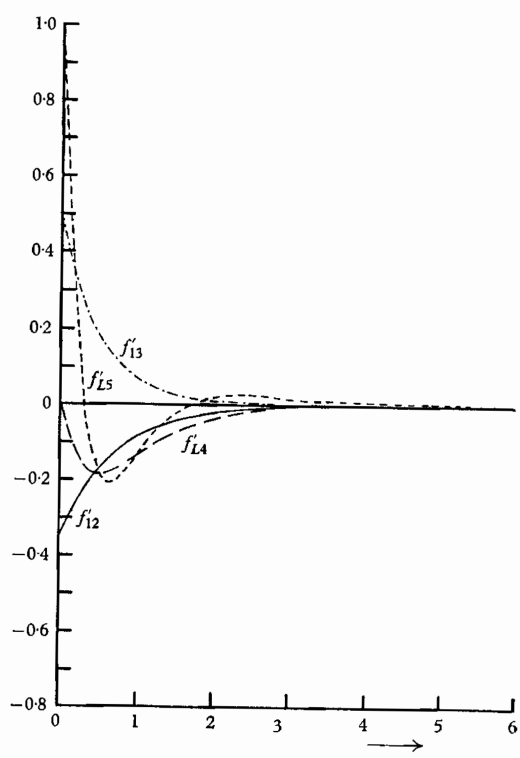

FIGURE 8. The functions $f'_{12}(y)$, $f'_{13}(y)$, $f'_{L4}(y)$, $f'_{L5}(y)$.

solution near $x = 0$, and cannot be attained without it. We know, of course, that $f_{14}(0) = 0$, and we can determine $f_{14}(\infty)$, by expanding $\phi_1(x)$ in powers of $\bar{x}$ (the value of $f_{14}(\infty)$ is approximately 0·125 and is marked on figure 7), but at this stage it is impossible to go any further.

Equation (5.1) for Γ_1 can also be expressed in terms of $\bar{x}$, and becomes

$$y\Gamma_{1yy} + y\Gamma_{1y} + (1-\bar{x})\,\Gamma_{1\bar{x}} = -\tfrac{1}{2}e^{-y}\psi_{1\bar{x}} \tag{6.5}$$

and we can set up an expansion for Γ_1 exactly similar to that for ψ_1 in (6.4), thus

$$\Gamma_1 = \bar{x}^2 g_{12}(y) + \bar{x}^3 g_{13}(y) + \bar{x}^4 g_{14}(y) + \ldots + \log \bar{x}[\bar{x}^4 g_{L4}(y) + \bar{x}^5 g_{L5}(y) + \ldots]. \tag{6.6}$$

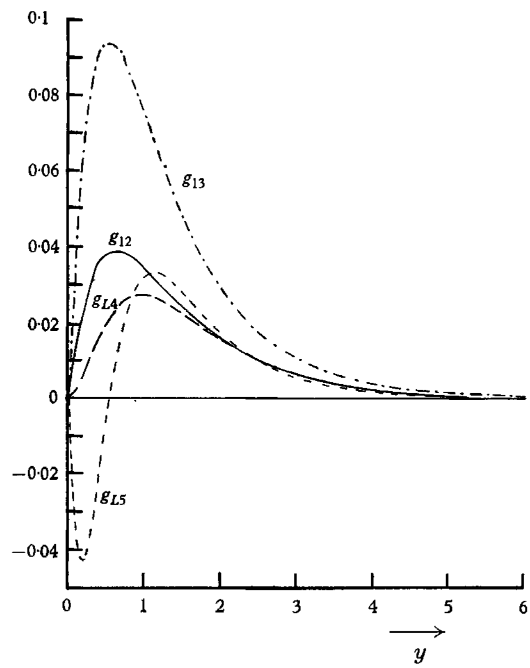

FIGURE 9. The leading terms of the first-order circulation expansion near $x = 1$: $g_{12}(y)$, $g_{13}(y)$, $g_{L4}(y)$, $g_{L5}(y)$.

Substituting (6.6) and (6.4) into (6.5), and equating like powers of $\bar{x}$, we obtain a series of equations which determine the functions g in terms of the functions f: for instance, $g_{12} = -\frac{1}{2}e^{-y}f_{12}$. The functions g_{12}, g_{13}, g_{L4}, g_{L5} are plotted against y in figure 9. The function g_{14} depends on f_{14}, and the indeterminacy in the latter affects the expansion for Γ_1 as well as that for ψ_1.

7. On the second-order solution

In the last three sections we have investigated in considerable detail the solutions of the equations to the first-order in K. The procedure for extending them to higher-orders in K is quite straightforward, but the results are less illuminating because in general it is not possible to perform the double integration of the equation for ψ_n, which in the case $n = 1$ led to a complete determination of the function $\phi_1(x)$ describing the radial inflow to the core. Consider the term of order K^2 in equation (3.4), which may be written

$$F^2(x)\,y\psi_{2yyyy}+F^2(x)\,(y+2)\,\psi_{2yyy}-x[F^2(x)\,\psi_2]_{xyy}$$
$$=\tfrac{1}{2}F^2(x)\,\psi_{1x}\psi_{1yyy}-\tfrac{1}{2}\psi_{1y}[F^2(x)\,\psi_1]_{xyy}-\frac{F'(x)}{8y}(\Gamma_0\Gamma_1)_y-\frac{F(x)}{8y^2}\Gamma_0\Gamma_{1x}. \quad (7.1)$$

There is no way of integrating (7.1) so as to leave an equation for the second-order inflow function $\phi_2(x)$, but there is nothing to prevent us from expanding ψ_2 in powers of x (as in (5.2) for ψ_1), and substituting the appropriate series into (7.1) as it stands, without prior integration. For instance, if the leading term for ψ_2 in powers of x is $x\psi_{21}(y)$, the equation for ψ_{21}, from (7.1) and the expansions of § 5, is

$$y\psi_{21}^{\mathrm{iv}}+(y+2)\,\psi_{21}'''-\psi_{21}''=\tfrac{1}{2}(\psi_{11}\psi_{11}'''-\psi_{11}'\psi_{11}'')$$
$$-(1/4a_0y^2)\,\{a_1y[e^{-y}\Gamma_{10}+(1-e^{-y})\,\Gamma_{10}']+(1-e^{-y})\,\Gamma_{12}\}. \quad (7.2)$$

Equation (7.2) has been integrated numerically, with boundary conditions $\psi_{21}(0)=\psi_{21}'(\infty)=0$, and the functions $\psi_{21}(y)$ and $\psi_{21}'(y)$ are plotted in figures 4 and 5, for comparison with the first-order solutions. The quantity $x\psi_{21}(\infty)$ is the leading term (in powers of x) of $\phi_2(x)$; the value of $\psi_{21}(\infty)$ is approximately 0·0031, which is small compared with $|\psi_{11}(\infty)|\approx 0{\cdot}0757$.

It is interesting to note that if we try to expand ψ_2 near the *sink*, in powers of $\bar{x}$, we are baulked from the start, because the leading term of the expansion turns out to be of the form $\bar{x}^4f_{24}(y)$, where f_{24} is an unknown function like f_{14}. Also, because we do not know the form of $\phi_2(x)$, we cannot even calculate $f_{24}(\infty)$.

We can similarly look at the K^2 term in (3.3), and obtain an equation for Γ_2 which may be attacked only by an expansion in powers of x. If the leading term of this expansion is $\Gamma_{20}(y)$, then Γ_{20} satisfies an equation, similar to (5.7), which has been integrated numerically and its solution plotted in figure 6. The leading term in the small $\bar{x}$ expansion for Γ_2 has the form $\bar{x}^4g_{24}(y)$, which is indeterminate because it depends on $f_{24}(y)$. The only way to derive further information about the second- and higher-order solutions is to continue with the expansions in powers of x, a straightforward but laborious process.

8. Discussion

The principal results of this paper are twofold. There is first of all the demonstration of how the presence of a narrow viscous vortex in general modifies the meridional component of the irrotational flow outside it, inducing a radial inflow to (or outflow from) the core, even for small values of the interaction parameter K (§ 4). That this is consistent with observation of vortices with large values of K has already been remarked. The only exceptions to this result are flows in which the axial stretching of vortex lines is uniform (i.e. the function $F(\xi)$ is a constant), when the meridional flow is undisturbed even inside the core, and the axial vorticity is itself independent of the axial co-ordinate.

The second achievement of the paper is the development of a method for the detailed calculation of the flow within the vortex core, at least near the stagnation point, for small values of K. This was applied to the particular sink-induced vortex defined by the function $F(\xi)$ of (2.11), but it should again be emphasized that the method is valid for all even functions $F(\xi)$ which are bounded at $\xi = 0$. The expansion of the flow variables near the stagnation point (small $x = \xi$) is unique and completely determined (§§ 5 and 7), while only the leading terms of the expansion near the sink (small $\bar{x} = 1-\xi$) can be calculated (§ 6). The others depend on functions which are presumably determined by conditions far from the sink. In any case, the second expansion has little relevance from a practical point of view, because it is precisely in the region of small $\bar{x}$ that the effects of finite sink size and of rigid boundaries will be dominant.

The actual velocity distributions in the vortex core, and hence a physical description of the flow, for small K, can be derived, both near the stagnation point and near the sink, from the functions plotted in figures 4–9. The meridional flow field of order K is best discussed in terms of the first-order contribution to the axial velocity distribution,

$$w_1 = -(Q/h^2)\,F(x)\,\psi_{1y}$$

(from (2.3), (1.5) and (3.2)). Indeed, since the leading terms in each expansion of ψ_{1y} (ψ'_{11}, ψ'_{13} near the stagnation point, and f'_{12}, f'_{13}, near the sink) are one-signed, monotonic functions of y, the behaviour of the axial velocity on the axis ($y = 0$) is a good indication of its behaviour elsewhere in the core. Near the stagnation point we have

$$w_1|_{y=0} = (Q/h^2)\,a_0\{-x\psi'_{11}(0) - x^3[a_1\psi'_{11}(0) + \psi'_{13}(0)] + O(x^5)\}$$
$$\approx (Q/h^2)\,\{0{\cdot}100x - 0{\cdot}007x^3 + O(x^5)\} \quad (8.1)$$

from equations (5.2) to (5.4) and figure 5; and near the sink we have

$$w_1|_{y=0} = (Q/h^2)\,\tfrac{1}{4}\{-f'_{12}(0) - \bar{x}[f'_{12}(0) + f'_{13}(0)] - \bar{x}^2 \log \bar{x} f'_{L4}(0) + O(\bar{x}^2)\}$$
$$\approx (Q/h^2)\,\{0{\cdot}087 - 0{\cdot}038\bar{x} + O(\bar{x}^2)\} \quad (8.2)$$

from equations (6.2) to (6.4) and figure 8. Note that both the zero and the first-order contributions to the axial velocity on the axis (from (3.5) and either (8.1) or (8.2)) increase with x for all x in $(0, 1)$, which exemplifies the statement of Batchelor (1964, § 2) that the axial velocity in a vortex increases as the core radius $\Delta(z)$, given in our case by (3.9), decreases. There is no tendency for the axial velocity to become negative anywhere, and hence no tendency for a multi-celled vortex to develop.

One expects the circulation in the vortex core to be intensified when there is an inflow, and weakened when there is an outflow, by conservation of angular momentum. The first-order circulation field near the stagnation point (see figure 6) verifies this expectation: Γ_{10} is positive, corresponding to the negative ψ_{11} (inflow), and Γ_{12} is negative, indicating a smaller intensification as the inflow decreases with x (ψ_{13} is positive). The situation near the sink seems to be anomalous, since the functions g are mostly positive (figure 9), which implies intensification of the vortex, despite the net outflow in that region. This is presumably a consequence of the geometry of the flow, in that the core radius is forced to go to zero at the sink.

We have no rigorous proof of convergence of the small x expansion, but we may note that the two terms given in (8.1) for the axial velocity on the axis agree at $x = 1$ with the small $\bar{x}$ expansion (8.2) to within 7 % (the two expressions are equal at $x \approx 0{\cdot}86$). Also, we know that the series obtained by letting y tend to infinity in the expansion (5.2) for $\psi_1(x, y)$ converges to $\phi_1(x)$, so that if we assume the later functions (ψ_{15}, etc.) to be as well-behaved as ψ_{11} and ψ_{13}, we have a good indication that the full expansion converges. What is more, it seems probable that the first two terms of this expansion yield a good representation of the flow field for quite large values of x. Not only is there the noted agreement between (8.1) and (8.2), but also, if we expand $\phi_1'(x)$ in powers of x, and plot just the first two terms on figure 2, we see acceptable agreement for values of x less than about 0·4.

One unexpected feature of the results is the fact that the function $\psi_{21}(y)$ is positive. This indicates that, as K is increased, the inflow near the stagnation point is *reduced*, whereas we would expect the effect to increase as K increases. However, the subsequent terms ψ_{23}, etc., may be negative, which would tend to make the inflow more uniform with x, in agreement with observation.

The important fact about the second-order functions ψ_{21} and Γ_{20} is that they are significantly smaller in magnitude than the corresponding first-order functions (figures 4, 5, 6), which are in turn small compared with the zero-order solution

$$\left|\frac{\psi_{21}(\infty)}{\psi_{11}(\infty)}\right| \approx 0{\cdot}04, \quad \left|\frac{\psi_{11}(\infty)}{1}\right| \approx 0{\cdot}08.$$

This demonstrates that our small K expansion is more than a mere mathematical exercise, but is a useful asymptotic expansion of the complete solution, yielding valid results for values of K which are not vanishingly small.

My thanks are due to Professor G. K. Batchelor for originally suggesting this problem while he was my research supervisor at Cambridge University, and to numerous people (especially Dr W. S. Lewellen) for the discussions which eventually set me on the way to solving it. The work was completed while I was at the Department of Mechanics, Johns Hopkins University. I am grateful also to the Johns Hopkins Homewood Computing Center for the use of their IBM 7094 computer, and to both the Science Research Council of Great Britain and the U.S. Office of Naval Research (Contract Nonr 4010(02)) for partial financial support during the course of this work.

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